Solution to Test 1, MMAT5000

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1. (i)(2 marks) Let A be a non-empty subset of $\mathbb{R}$. A real number $t = \inf A$ if and only if the following two conditions are satisfied:

   a) $t \leq a$ for all $a \in A$;

   b) For each $v \in \mathbb{R}$ satisfying $t < v$, there exists $a_v \in A$ such that $a_v < v$.

   (ii)(4 marks) Let $A = \{x \in \mathbb{R} : 0 < \sin(x^{-1}) < \frac{\sqrt{3}}{2}\}$, then $\inf A = -\pi^{-1}$. We need to show that $-\pi^{-1}$ satisfies the two conditions listed in (i). In fact, for any $x \in A$, we have

   $$x^{-1} \in \left(2k\pi, (2k + \frac{1}{3})\pi\right) \bigcup \left((2k + \frac{2}{3})\pi, (2k+1)\pi\right), \quad k \in \mathbb{Z}$$

   i.e.

   $$x \in \left(\frac{3}{6k+1}\pi^{-1}, \frac{1}{2k}\pi^{-1}\right) \bigcup \left(\frac{1}{2k+1}\pi^{-1}, \frac{3}{6k+2}\pi^{-1}\right), \quad k \in \mathbb{Z}.$$

   Notice that

   $$\frac{3}{6k+1}\pi^{-1} > \frac{1}{2k+1}\pi^{-1} \geq -\pi^{-1}, \quad \text{for all } k \in \mathbb{Z},$$

   so $-\pi^{-1}$ is a lower bound of A, i.e. Condition (a) in (i) holds.
   Next, for each $v \in \mathbb{R}$ with $-\pi^{-1} < v$,

   – If $v \geq 0$, then we can find $-\frac{4}{5}\pi^{-1} \in A$ such that $-\frac{4}{5}\pi^{-1} < 0 \leq v$;

   – If $v < 0$, then $v^{-1} < -\pi$. There exists $t_v \in \left(-\frac{4}{3}\pi, -\pi\right)$ such that $v^{-1} < t_v < -\pi$. Set $a_v = t_v^{-1}$, then we have $a_v \in A$ and $a_v < v$.

   Condition (b) holds too, so $\inf A = -\pi^{-1}$.

2. (6 marks)

   – If $c = 3$, then $a_n = 3$ for all $n \geq 1$. We will show it by mathematical induction: i) $a_1 = 3$; ii) Assume that $a_k = 3$ for all $k \geq 1$; iii) Then $a_{k+1} = \sqrt{3a_k} = 3$.
   So $\lim_{n\to\infty} a_n = 3$.

   – If $0 < c < 3$, then $0 < a_n < 3$ for all $n \geq 1$. We will show it by mathematical induction: i) $a_1 = c \in (0,3)$; ii) Assume that $a_k \in (0,3)$ for all $k \geq 1$; iii) Then $a_{k+1} = \sqrt{3a_k} \in (0,3)$.
   Moreover, for all $n \geq 1$, we have

   $$a_{n+1} - a_n = \sqrt{3a_n} - a_n = \sqrt{a_n}(\sqrt{3} - \sqrt{a_n}) > 0.$$

   So $\{a_n\}_{n\in\mathbb{N}}$ is an increasing sequence and has an upper bound, and hence converges. Set $l = \lim_{n\to\infty} a_n$, then we have

   $$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n},$$

i.e.

$$l = \sqrt{3l},$$

which implies that $l = 3$. (Here we rejected the case $l = 0$, since we notice that $a_n \geq c > 0$ for all $n \geq 1$.)

- If $c > 3$, then $a_n > 3$ for all $n \geq 1$. We will show it by mathematical induction: i) $a_1 = c > 3$; ii) Assume that $a_k > 3$ for all $k \geq 1$; iii) Then $a_{k+1} = \sqrt{3a_k} > 3$. Moreover, for all $n \geq 1$, we have

$$a_{n+1} - a_n = \sqrt{3a_n} - a_n = \sqrt{a_n}(\sqrt{3} - \sqrt{a_n}) < 0.$$

So $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence and has a lower bound, and hence converges. Set $m = \lim\limits_{n\to\infty} a_n$, then we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n},$$

i.e.

$$m = \sqrt{3m},$$

which implies that $m = 3$. (Here we rejected the case $m = 0$, since we notice that $a_n > 3 > 0$ for all $n \geq 1$.)

Therefore, $\lim\limits_{n\to\infty} a_n = 3$.

3. (i)(5 marks)First, we claim that

$$0 \leq y_2 \leq y_4 \leq \cdots \leq y_{2n} \leq y_{2n+2} \leq \cdots \leq y_{2n+1} \leq y_{2n-1} \leq \cdots \leq y_3 \leq y_1 = x, \quad n \geq 1.$$

It suffices to show that for all $n \geq 1$:

$$y_n \in [0, x],\ y_{2n} \leq y_{2n+2},\ y_{2n+1} \leq y_{2n-1},\ y_{2n} \leq y_{2n-1}.$$

I) $y_1 - y_2 = \frac{1}{2}y_1^2 \geq 0,\ y_1 - y_3 = \frac{1}{2}y_2^2 \geq 0,\ y_2 - y_4 = \frac{1}{2}(y_3^2 - y_1^2) \geq 0;$
II)Assume that it holds for $n = k$, i.e. $y_k \in [0, x],\ y_{2k} \leq y_{2k+2},\ y_{2k+1} \leq y_{2k-1},\ y_{2k} \leq y_{2k-1},\quad k \geq 1;$
III)For $n = k+1$, we have $y_{2k+1} - y_{2k} = \frac{1}{2}(y_{2k-1}^2 - y_{2k}^2) \geq 0,\ y_{2k+1} - y_{2k+2} = \frac{1}{2}(y_{2k+1}^2 - y_{2k}^2) \geq 0,\ y_{2k+1} - y_{2k+3} = \frac{1}{2}(y_{2k+2}^2 - y_{2k}^2) \geq 0,\ y_{2k+2} - y_{2k+4} = \frac{1}{2}(y_{2k+3}^2 - y_{2k+1}^2) \leq 0.$
Second, for any $p > m \geq 4$,

$$\begin{aligned}
|y_p - y_m| &= \frac{1}{2}|y_{p-1}^2 - y_{m-1}^2| \\
&= \frac{1}{2}|y_{p-1} + y_{m-1}| \cdot |y_{p-1} - y_{m-1}| \\
&\leq y_3 \cdot |y_{p-1} - y_{m-1}| \\
&= y_3 \cdot \frac{1}{2}|y_{p-2}^2 - y_{m-2}^2| \\
&= y_3 \cdot \frac{1}{2}|y_{p-2} + y_{m-2}| \cdot |y_{p-2} - y_{m-2}| \\
&\leq y_3^2 \cdot |y_{p-2} - y_{m-2}| \\
&\quad \cdots \quad \cdots \\
&\leq y_3^{m-2} \cdot |y_{p-m+2} - y_2| \\
&\leq 2y_3^m,
\end{aligned}$$

where we used the fact that $y_n \leq y_3$ for all $n \geq 2$.
Direct calculations gives

$$y_3 = x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 \leq \frac{7}{8}, \quad x \in [0,1].$$

Finally, for any $\varepsilon > 0$, there exists $N = \left[\frac{\ln 2 - \ln \varepsilon}{ln8 - ln7}\right] + 1$ such that for all $p > m > N$, we have

$$|y_p - y_m| \leq 2y_3^m \leq 2(\frac{7}{8})^N < \varepsilon.$$

Therefore, $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy.
(ii)(1 marks) If $\varepsilon = 0.1$, then our $N = 23$.